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Exact solutions of triple-order time-fractional differential equations for anomalous relaxation and diffusion I: The accelerating case

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ABSTRACT

In recent years the interest around the study of anomalous relaxation and diffusion processes is increased due to their importance in several natural phenomena. Moreover, a further generalization has been developed by introducing time-fractional differentiation of distributed order which ranges between 0 and 1. We refer to accelerating processes when the driving power law has a changing-in-time exponent whose modulus tends from less than 1 to 1, and to decelerating processes when such an exponent modulus decreases in time moving away from the linear behaviour. Accelerating processes are modelled by a time-fractional derivative in the Riemann–Liouville sense, while decelerating processes by a time-fractional derivative in the Caputo sense. Here the focus is on the accelerating case while the decelerating one is considered in the companion paper. After a short reminder about the derivation of the fundamental solution for a general distribution of time-derivative orders, we consider in detail the triple-order case for both accelerating relaxation and accelerating diffusion processes and the exact results are derived in terms of an infinite series of *H*-functions. The method adopted is new and it makes use of certain properties of the generalized Mittag-Leffler function and the *H*-function, moreover it provides an elegant generalization of the method introduced by Langlands (2006) [T.A.M. Langlands, Physica A 367 (2006) 136] to study the double-order case of accelerating diffusion processes.

1. Introduction

Anomalous dynamics is frequently met in processes through complex and/or disordered media, e.g. dispersion in plasmas [1] or diffusion with obstacles [2] or binding [3], self-diffusion of surfactant molecules [4] or protein movements [5,6], and light scattering in a cold atomic cloud [7]. A useful mathematical tool for physical investigation and description of such phenomena is fractional calculus, see for example Refs. [8–13] for anomalous relaxation and Refs. [14–20] for anomalous diffusion. Recently, the extension of fractional differential equations to distributed-order fractional differential equations has permitted to describe also processes whose scaling law changes in time [21–23,12,24–28]. However an early idea of the time-fractional derivative of distributed order was proposed in 1969 by Caputo [29], and later re-proposed by Caputo himself [30,31] and Bagley and Torvik [32,33]. In particular, when the time-fractional derivative operator is opportunely chosen, it permits to model phenomena whose driving power law can be expressed by a growing-in-time exponent modulus that from less than 1 tends to 1, or an exponent modulus that decreases in time moving away from the linear behaviour, and we refer to them as accelerating and decelerating processes, respectively [34,35]. The accelerating processes are modelled by a

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Riemann–Liouville (R–L) time-fractional derivative while the decelerating processes by a Caputo (C) time-fractional derivative. However, when single-order fractional differential equations are considered the two forms (R–L) and (C) are equivalent. The present paper is focused on the accelerating case while the decelerating case is considered in the companion paper [36].

A double-order time-fractional diffusion equation has been exactly solved by Langlands [37,38]. The main object of the present article is to further include one more fractional time derivative and investigate its solution by the application of the generalized Mittag-Leffler function. Then the triple-order time-fractional differential equations considered in the present paper are:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\lambda \left[P_t D^{1-\alpha} + Q_t D^{1-\beta} + T_t D^{1-\gamma} \right] u(t), \quad t \in \mathbf{R}_0^+, \tag{1}$$

$$\frac{\partial u}{\partial t} = \left[P_t D^{1-\alpha} + Q_t D^{1-\beta} + T_t D^{1-\gamma} \right] \frac{\partial^2}{\partial x^2} u(x,t), \quad t \in \mathbf{R}_0^+,\ x \in \mathbf{R}, \tag{2}$$

with $0 < \alpha < \beta < \gamma < 1$, where λ is a positive constant and ${}_tD^{\nu}$ is the Riemann–Liouville time-fractional derivative operator of order $\nu > 0$ [39,40], which for a sufficiently well-behaved function $f(t)$ is defined as

$${}_tD^{\nu} f(t) = \begin{cases} \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} \left[\dfrac{1}{\Gamma(m-\nu)} \displaystyle\int_0^t \frac{f(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\nu+1-m}} \right], & m-1 < \nu < m, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), & \nu = m. \end{cases} \tag{3}$$

The rest of the paper is organized as follows. In Section 2 the fundamental solution for a general distribution of time-derivative orders is recalled together with general considerations which motivate the name *accelerating*. In Section 3 the exact solutions for both triple-order time-fractional relaxation and diffusion equations are obtained using a new method based on certain properties of Mittag-Leffler and H-functions. Concluding remarks are given in Section 4.

2. Accelerating time-fractional relaxation and diffusion

2.1. Accelerating time-fractional relaxation equation

The equation of accelerating time-fractional relaxation is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\lambda \int_0^1 p(\nu)\, {}_tD^{1-\nu} u(t)\,\mathrm{d}\nu, \quad u(0^+) = 1,\ \lambda \in \mathbf{R}^+,\ t \in \mathbf{R}_0^+, \tag{4}$$

where $p(\nu) \geq 0$ is the weight function of the fractional order derivative, which is taken normalized; i.e. $\int_0^1 p(\nu)\,\mathrm{d}\nu = 1$. A general theoretical analysis of time-fractional relaxation of distributed order can be found in Ref. [12]. Let the Laplace transform for a generic function $w(t)$ be defined as:

$$\mathcal{L}\{w(t); s\} = \widetilde{w}(s) := \int_0^{+\infty} \mathrm{e}^{-st} w(t)\,\mathrm{d}t, \quad s \in \mathbf{C}.$$

We recall that, if the limiting values of the k-integer derivatives $w^{(k)}(0^+)$ for $k = 0, 1, 2, \ldots$ are finite, the Laplace transform of the Riemann–Liouville fractional derivative of non-integer order ν defined in the first line of (3) is $\mathcal{L}\{{}_tD^{\nu} w(t); s\} = s^{\nu}\widetilde{w}(s)$, see Ref. [41, formula (1.15)] or Ref. [42, Ch. 1 formula (1.29)]. Then, applying the Laplace transform to (4) gives

$$\widetilde{u}(s) = \frac{1}{s + \lambda A(s)}, \tag{5}$$

where

$$A(s) = s \int_0^1 p(\nu) s^{-\nu}\,\mathrm{d}\nu. \tag{6}$$

In the case of singular order, i.e. $p(\nu) = \delta(\nu - \nu_0)$, it turns out that (5) is the same as in the decelerating case [36] and both provide the same result of the simple fractional relaxation.

By inverting the Laplace transform in (5) the fundamental solution $u(t)$ of the accelerating time-fractional relaxation equation of distributed order (4) is obtained. Solution $u(t)$ can be represented by the following Laplace inversion formula [43,44, pp. 31–33], referred also to as the Titchmarsh theorem as for example in Refs. [22,23,12,24],

$$u(t) = -\frac{1}{\pi} \int_0^{\infty} \mathrm{e}^{-rt} \mathrm{Im}\left\{ \widetilde{u}\left(r\mathrm{e}^{\mathrm{i}\pi}\right) \right\} \mathrm{d}r, \tag{7}$$

and the expression of $-\mathrm{Im}\{1/[s + \lambda A(s)]\}$ is needed along the ray $s = r\,\mathrm{e}^{\mathrm{i}\pi}$ with $r > 0$ (the branch cut of the function $s^{-\nu}$).

When the function $A(s)$ defined in (6) is rewritten as

$$A\left(r\,\mathrm{e}^{\mathrm{i}\pi}\right) = \rho\cos(\pi\gamma) + \mathrm{i}\rho\sin(\pi\gamma), \tag{8}$$

where

$$\begin{cases} \rho = \rho(r) = \left|A\left(r\,\mathrm{e}^{\mathrm{i}\pi}\right)\right|, \\ \gamma = \gamma(r) = \dfrac{1}{\pi}\arg\left[A\left(r\,\mathrm{e}^{\mathrm{i}\pi}\right)\right], \end{cases}$$

then the fundamental solution of (4) turns out to be

$$u(t) = \int_0^\infty \mathrm{e}^{-rt} H(r,\lambda)\,\mathrm{d}r, \tag{9}$$

with

$$H(r,\lambda) = \frac{1}{\pi}\frac{\lambda\rho\sin(\pi\gamma)}{r^2 - 2\lambda r\rho\cos(\pi\gamma) + \lambda^2\rho^2} \geq 0. \tag{10}$$

Since $H(r,\lambda)$ is a non-negative function of r for any $\lambda \in \mathbf{R}^+$, the fundamental solution $u(t)$ is completely monotonic.

In order to highlight the accelerating property, the asymptotic expressions for $t \to 0$ and $t \to \infty$ of the fundamental solution $u(t)$ are provided. According to the Tauberian theory of Laplace transforms, the asymptotic behaviour of a function $w(t)$ near $t = \infty$ and $t = 0$ is (formally) obtained from the asymptotic behaviour of its Laplace transform $\widetilde{w}(s)$ for $s \to 0^+$ and for $s \to +\infty$, respectively. The asymptotic representations of (5) are [12]

$$\widetilde{u}(s) \sim \begin{cases} \dfrac{1}{\lambda A(s)}, & s \to 0^+, \text{ when } A(s)/s \gg \lambda, \\ \dfrac{1}{s}\left[1 - \lambda\dfrac{A(s)}{s}\right], & s \to +\infty, \text{ when } A(s)/s \ll 1/\lambda. \end{cases}$$

As it is shown for a double-order case in Ref. [12], i.e. $p(\nu) = p_1\delta(\nu - \nu_1) + p_2\delta(\nu - \nu_2)$, $0 < \nu_1 < \nu_2 \leq 1$,

$$u(t) \sim \begin{cases} 1 - \lambda p_1\dfrac{t^{\nu_1}}{\Gamma(1+\nu_1)}, & t \to 0^+, \\ \dfrac{1}{\lambda p_2}\dfrac{t^{-\nu_2}}{\Gamma(1-\nu_2)}, & t \to +\infty, \end{cases}$$

from which follows the name *accelerating* relaxation, because, remembering that $0 < \nu_1 < \nu_2 \leq 1$, the modulus of the exponent of the relaxation power law increases towards 1 when the time increases.

2.2. Accelerating time-fractional diffusion equation

The equation of accelerating time-fractional diffusion is

$$\frac{\partial u}{\partial t} = \int_0^1 p(\nu)\,{}_tD^{1-\nu}\left[\frac{\partial^2}{\partial x^2}u(x,t)\right]\mathrm{d}\nu, \quad u(x,0^+) = \delta(x),\ t \in \mathbf{R}_0^+,\ x \in \mathbf{R}, \tag{11}$$

where the weight function $p(\nu)$ has the same properties as in Section 2.1. For a general study of accelerating time-fractional differential equations of distributed order, see Refs. [22,24].

The fundamental solution of the time-fractional diffusion equation of distributed order (11) can be obtained by applying in sequence the Fourier and Laplace transforms. Let the Fourier transform for a generic function $v(x)$ be defined as:

$$\mathcal{F}\{v(x);\kappa\} = \widehat{v}(\kappa) := \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\kappa x} v(x)\,\mathrm{d}x, \quad \kappa \in \mathbf{R},$$

then, in the Fourier–Laplace domain, noting that $\widehat{\delta}(\kappa) \equiv 1$, Eq. (11) becomes

$$\widehat{\widetilde{u}}(\kappa,s) - 1 = -\kappa^2\left[\int_0^\infty p(\nu)s^{1-\nu}\,\mathrm{d}\nu\right]\widehat{\widetilde{u}}(\kappa,s), \tag{12}$$

and using definition (6) of function $A(s)$

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{1}{s + \kappa^2 A(s)}. \tag{13}$$

In the particular case of time-fractional diffusion of single order, i.e. $p(\nu) = \delta(\nu - \nu_0)$ $(0 < \nu_0 \leq 1)$, accelerating and decelerating processes [36] provide the same result.

In order to determine the fundamental solutions $u(x,t)$, Eq. (13) must be inverted with respect to both Fourier and Laplace transformation. However, the inversion of Laplace transformation has been obtained for the relaxation equation and then

we have to invert the remaining Fourier transform. In fact, setting $\lambda = \kappa^2$, Eq. (13) is equal to (5). Then, since $u(x, t)$ is symmetric in x, using (9) the fundamental solution of (11) is given by

$$u(x, t) = \frac{1}{\pi} \int_0^{+\infty} \cos(\kappa x) \left\{ \int_0^{\infty} \mathrm{e}^{-rt} H(r, \kappa^2)\, \mathrm{d}r \right\} \mathrm{d}\kappa, \tag{14}$$

where $H(r, \kappa^2)$ is the same as defined in (10). When the Caputo fractional derivative is used, the Fourier integral (14) has been carried out by Mainardi and Pagnini [25] by the method of the Mellin transform. However, in Refs. [22,24] the relationships are derived between the fundamental solutions of distributed order time-fractional diffusion equations with Caputo and Riemann–Liouville fractional derivatives. Finally, the fundamental solution $u(x, t)$ in (14), for $x \geq 0$, reads

$$u(x, t) = \frac{1}{2\pi x} \int_0^{\infty} \frac{\mathrm{e}^{-rt}}{r} F_*(\rho_*^{1/2} x)\, \mathrm{d}r, \tag{15}$$

where

$$F_*(\rho_*^{1/2} x) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \Gamma(1 - s) \sin(\pi \gamma_* s/2) (\rho_*^{1/2} x)^s \mathrm{d}s, \tag{16}$$

with $\rho_*(r) = r/\rho(r)$ and $\gamma_*(r) = 1 - \gamma(r)$. The series expansion of F_* yields the required solution

$$u(x, t) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \varphi_k(t), \quad x \geq 0, \tag{17}$$

with

$$\varphi_k(t) = \int_0^{\infty} \frac{\mathrm{e}^{-rt}}{r} \sin[\pi \gamma_* (k + 1)/2]\, \rho_*^{(k+1)/2}\, \mathrm{d}r. \tag{18}$$

An alternative representation of F_* reads [22,24]

$$\begin{aligned} F_*(\rho_*^{1/2} x) &= \mathrm{Im}\{\rho_*^{1/2} x\, \mathrm{e}^{\mathrm{i}\pi\gamma_*/2} \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\pi\gamma_*/2} \rho_*^{1/2} x}\} \\ &= \rho_*^{1/2} x\, \mathrm{e}^{-\rho_*^{1/2} x \cos(\pi\gamma_*/2)} \sin[\pi\gamma_*/2 - \rho^{1/2} x \sin(\pi\gamma_*/2)], \end{aligned} \tag{19}$$

which gives a more convenient representation in comparison to the series (17) to plot the solution from the numerical point of view, when it is inserted in (15).

To classify the type of diffusion as accelerating the behaviour in time of the variance $\sigma^2(t)$ defined as

$$\sigma^2(t) := \int_{-\infty}^{+\infty} x^2 u(x, t)\, \mathrm{d}x, \tag{20}$$

which is related to the Fourier transform by

$$\sigma^2(t) = -\frac{\partial^2}{\partial \kappa^2} \widehat{u}(\kappa = 0, t), \tag{21}$$

is relevant. This means that we have to invert only the Laplace transform and then take into account the Fourier transform for κ near zero. Finally, from (13) we have

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{1}{s} \left(1 - \kappa^2 \frac{A(s)}{s} + \cdots \right),$$

and then

$$\widetilde{\sigma^2}(s) = -\frac{\partial^2}{\partial \kappa^2} \widehat{\widetilde{u}}(\kappa = 0, s) = \frac{2A(s)}{s^2}.$$

As for the relaxation equation, considering the double-order case gives [22,24]

$$\sigma^2(t) \sim \begin{cases} 2p_1 \dfrac{t^{\nu_1}}{\Gamma(1 + \nu_1)}, & t \to 0^+, \\ 2p_2 \dfrac{t^{\nu_2}}{\Gamma(1 + \nu_2)}, & t \to +\infty, \end{cases}$$

where p_1, p_2, ν_1 and ν_2 are the same as for the double-order case considered for accelerating relaxation, and then the name *accelerating* diffusion follows, because, remembering that $0 < \nu_1 < \nu_2 \leq 1$, the exponent of the diffusion power law increases towards 1 when the time increases.

3. Solutions of triple-order accelerating relaxation and diffusion

In this section we present a new method to calculate the exact solution of the triple-order time-fractional differential equation of accelerating relaxation (1) and accelerating diffusion (2), which is based on the Prabhakar generalization of the Mittag-Leffler function, see Appendix B. In general, the self-similarity of solutions of the ordinary single-order

time-fractional equations is due to the unique derivative order, so that such self-similarity is lost in distributed cases. In particular, for discrete distributions of derivative orders, the number of time scales of solutions is equal to the number of derivative orders, so that in the case under consideration the time scales of solutions are three.

3.1. Triple-order accelerating relaxation

The triple-order time-fractional differential equation for accelerated anomalous relaxation (1) is derived by (4) with the weight function

$$p(\nu) = P\delta(\nu-\alpha) + Q\delta(\nu-\beta) + T\delta(\nu-\gamma), \tag{22}$$

with $P+Q+T=1$ by normalization, and the Laplace transformed solution corresponding to (5) is

$$\widetilde{u}(s) = \frac{s^{\alpha-1}}{s^\alpha+\lambda P}\,\frac{s^\alpha+\lambda P}{s^\alpha+\lambda P+\lambda Q s^{\alpha-\beta}+\lambda T s^{\alpha-\gamma}}. \tag{23}$$

Using the following formula

$$\frac{1}{1+\xi} = \sum_{m=0}^{\infty}(-1)^m\xi^m, \quad |\xi|<1, \tag{24}$$

the second fraction on the RHS of (23) turns out to be

$$\widetilde{u}(s) = \frac{s^{\alpha-1}}{s^\alpha+\lambda P}\sum_{m=0}^{\infty}\frac{(-1)^m\lambda^m Q^m(s^{\alpha-\beta}+T/Qs^{\alpha-\gamma})^m}{(s^\alpha+\lambda P)^m}. \tag{25}$$

Moreover, using the series representation

$$(a+\xi)^m = \sum_{\ell=0}^{m}\binom{m}{\ell}\xi^\ell a^{m-\ell}, \tag{26}$$

we have that

$$\left(s^{\alpha-\beta}+\frac{T}{Q}s^{\alpha-\gamma}\right)^m = \sum_{\ell=0}^{m}\binom{m}{\ell}\left(\frac{T}{Q}\right)^\ell s^{(\alpha-\beta)m+(\beta-\gamma)\ell}, \tag{27}$$

and the Laplace transform of the solution emerges to be

$$\widetilde{u}(s) = \sum_{m=0}^{\infty}\frac{(-1)^m\lambda^m Q^m}{(1+\lambda P s^{-\alpha})^{m+1}}\sum_{\ell=0}^{m}\binom{m}{\ell}\left(\frac{T}{Q}\right)^\ell s^{\beta(\ell-m)-\gamma\ell-1}. \tag{28}$$

Finally, from the Laplace transform pair (B.5), the fundamental solution of (1) is

$$u(t) = \sum_{m=0}^{\infty}(-1)^m\left[\lambda Q t^\beta\right]^m\sum_{\ell=0}^{m}\binom{m}{\ell}\left(\frac{T}{Q}\right)^\ell t^{(\gamma-\beta)\ell}E^{m+1}_{\alpha,\gamma\ell+\beta(m-\ell)+1}(-\lambda P t^\alpha), \tag{29}$$

which in terms of H-function (see Appendix A) is

$$u(t) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left[\lambda Q t^\beta\right]^m\times\sum_{\ell=0}^{m}\binom{m}{\ell}\left(\frac{T}{Q}\right)^\ell t^{(\gamma-\beta)\ell}H^{1,1}_{1,2}\left[\lambda P t^\alpha\,\middle|\,\begin{matrix}(-m,1)\\(0,1),[\beta(\ell-m)-\gamma\ell,\alpha]\end{matrix}\right], \tag{30}$$

where the H-function representation of the generalized Mittag-Leffler (B.7) is used.

It is worth noting here that in (29) the Prabhakar generalization of the Mittag-Leffler function $E^{m+1}_{\alpha,\gamma\ell+\beta(m-\ell)+1}(-\lambda P t^\alpha)$, which is defined in (B.3), can be expressed in terms of the mth-derivative of the two-parameter Mittag-Leffler function (B.2) by using formula (B.4).

3.1.1. Special cases

For $\beta=\gamma$, Eq. (1) turns out to be a double-order accelerating relaxation equation, and using (26) the above formula (30) reduces to

$$u(t) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left[\lambda(Q+T)t^\beta\right]^m H^{1,1}_{1,2}\left[\lambda P t^\alpha\,\middle|\,\begin{matrix}(-m,1)\\(0,1),(-\beta m,\alpha)\end{matrix}\right], \tag{31}$$

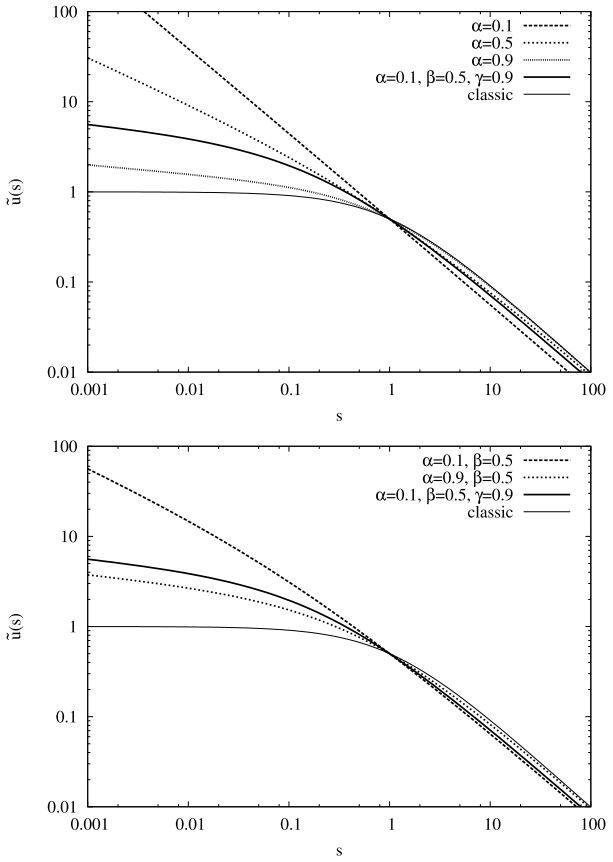

Fig. 1. Comparison of the frequency spectra of triple-order (5) and classic cases with those of single- (top) and double-order (bottom) cases when $\lambda = 1$. In the single-order case $\{P = 1, Q = T = 0, \alpha = 0.1, 0.5, 0.9\}$, in the double-order case $\{P = Q = 1/2, T = 0\}$ with $\{(\alpha = 0.1, \beta = 0.5); (\alpha = 0.5, \beta = 0.9)\}$ and in the triple-order case $\{P = Q = T = 1/3, \alpha = 0.1, \beta = 0.5, \gamma = 0.9\}$.

and for $\beta = \gamma = \alpha$, using H-function properties (A.4), (A.5) and the H-function representation of Mittag-Leffler function (B.7), formula (30) reduces to the simple anomalous relaxation solution

$$u(t) = H_{1,2}^{1,1}\left[\lambda(P+Q+T)t^{\alpha}\,\middle|\,\begin{matrix}(0,1)\\(0,1),(0,\alpha)\end{matrix}\right], \tag{32}$$

$$= E_{\alpha}[-\lambda(P+Q+T)t^{\alpha}], \tag{33}$$

originally derived by Mainardi [11].

The Laplace transform (5) can be seen also as the frequency spectrum of the fundamental solution of the distributed time-fractional accelerating relaxation. These spectra for single, double and triple order are plotted and compared in Fig. 1 together with the classical solution of the relaxation equation (i.e. single order equal to 1). It is observed that for large frequencies, which means small elapsed times, the behaviour in all considered cases is similar while for small frequencies, which means large elapsed times, the competition among the time-differentiation orders causes different behaviours. In particular, if the multi-order cases orders near to 1 are present, then their spectrum will be closer to that of the classic case than the single-order spectra with order much less than 1. On the other hand, because of the presence in the multi-order cases of orders much less than 1, their spectrum will be less close to the classic one than the single-order spectrum with order close to 1. The same type of behaviour is observed between triple- and double-order cases when the higher orders are compared.

3.2. Triple-order accelerating diffusion

As it has been noted in Section 2.2, the solutions of accelerating relaxation and of accelerating diffusion are related by $\lambda = \kappa^2$, where κ is the wavenumber variable in the Fourier domain. Then, applying this change to formula (29), the Fourier transform of the accelerating diffusion solution is

$$\widehat{u}(\kappa, t) = \sum_{m=0}^{\infty}(-1)^m\left[\kappa^2 Q t^{\beta}\right]^m \sum_{\ell=0}^{m}\binom{m}{\ell}\left(\frac{T}{Q}\right)^{\ell} t^{(\gamma-\beta)\ell} E_{\alpha,\gamma\ell+\beta(m-\ell)+1}^{m+1}(-\kappa^2 P t^{\alpha}), \tag{34}$$

or, in terms of the H-function, from formula (30)

$$\widehat{u}(\kappa,t)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left[\kappa^2Qt^{\beta}\right]^m\sum_{\ell=0}^{m}\binom{m}{\ell}\left(\frac{T}{Q}\right)^{\ell}t^{(\gamma-\beta)\ell}H_{1,2}^{1,1}\left[\kappa^2Pt^{\alpha}\,\middle|\,\begin{matrix}(-m,1)\\(0,1),(\beta(\ell-m)-\gamma\ell,\alpha)\end{matrix}\right]. \tag{35}$$

Thus on account of (35), we need to invert for each m in the expression

$$\widehat{g}_m(\kappa,t)=\kappa^{2m}H_{1,2}^{1,1}\left[\kappa^2Pt^{\alpha}\,\middle|\,\begin{matrix}(-m,1)\\(0,1),(\beta(\ell-m)-\gamma\ell,\alpha)\end{matrix}\right]. \tag{36}$$

It is well known that the Fourier transform of an even function can be written in terms of the cosine Fourier transform as

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\kappa x}f(\kappa)\,\mathrm{d}\kappa=\frac{1}{\pi}\int_{0}^{+\infty}f(\kappa)\cos(\kappa x)\,\mathrm{d}\kappa. \tag{37}$$

On employing (37) to invert (36) and using integral formula (A.8), we find that

$$g_m(x,t)=\frac{1}{|x|^{2m+1}}H_{3,3}^{2,1}\left[\frac{x^2}{Pt^{\alpha}}\,\middle|\,\begin{matrix}(1,1),(1+\gamma\ell-\beta(\ell-m),\alpha)(1+m,1)\\(2m+1,2)(1+m,1),(1+m,1)\end{matrix}\right], \tag{38}$$

which, by means of the definition of the H-function (A.1), can be written as the following Mellin–Barnes integral

$$g_m(x,t)=\frac{1}{|x|^{2m+1}}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\Gamma(s)\Gamma(2m+1-2s)\Gamma(1+m-s)}{\Gamma(s-m)\Gamma(1+m-s)\Gamma(1+\gamma\ell-\beta(\ell-m)-\alpha s)}\left(\frac{x^2}{Pt^{\alpha}}\right)^{s}\mathrm{d}s.$$

If we apply the rule $\Gamma(1+z)=z\Gamma(z)$ and the duplication formula $\Gamma(z)\Gamma(1/2+z)=\Gamma(2z)\sqrt{\pi}2^{1-2z}$, the above formula simplifies to

$$g_m(x,t)=\frac{2^{2m}}{\sqrt{\pi}|x|^{2m+1}}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\Gamma(s)\Gamma\left(m+\frac{1}{2}-s\right)\Gamma(1+m-s)}{\Gamma(s-m)\Gamma(1+\gamma\ell-\beta(\ell-m)-\alpha s)}\left(\frac{x^2}{4Pt^{\alpha}}\right)^{s}\mathrm{d}s, \tag{39}$$

which yields

$$g_m(x,t)=\frac{2^{2m}}{\sqrt{\pi}|x|^{2m+1}}H_{2,3}^{2,1}\left[\frac{x^2}{4Pt^{\alpha}}\,\middle|\,\begin{matrix}(1,1),(1+\gamma\ell-\beta(\ell-m),\alpha)\\(m+1/2,1)(1+m,1),(1+m,1)\end{matrix}\right]. \tag{40}$$

Now, the application of property (A.4) to absorb the expression $|x|^{-2m-1}$ inside the H-function in (40) gives the result

$$g_m(x,t)=\frac{(Pt^{\alpha})^{-m}}{\sqrt{4\pi Pt^{\alpha}}}H_{2,3}^{2,1}\left[\frac{x^2}{4Pt^{\alpha}}\,\middle|\,\begin{matrix}(1/2-m,1),(1-\alpha/2-\alpha m+\gamma\ell-\beta(\ell-m),\alpha)\\(0,1)(1/2,1),(1/2,1)\end{matrix}\right].$$

Thus from (35), we obtain the desired solution of Eq. (2) in an infinite domain

$$\begin{aligned}u(x,t)=&\frac{1}{\sqrt{4\pi Pt^{\alpha}}}\sum_{m=0}^{\infty}(Pt^{\alpha})^{-m}\frac{(-Qt^{\beta})^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}\left(\frac{T}{Q}\right)^{\ell}t^{\ell(\gamma-\beta)}\\&\times H_{2,3}^{2,1}\left[\frac{x^2}{4Pt^{\alpha}}\,\middle|\,\begin{matrix}(1/2-m,1),(1-\alpha/2-\alpha m+\gamma\ell-\beta(\ell-m),\alpha)\\(0,1)(1/2,1),(1/2,1)\end{matrix}\right].\end{aligned} \tag{41}$$

3.2.1. Special cases

When $\beta=\gamma$, Eq. (2) turns out to be a double-order accelerating diffusion equation, and the above result (41), using formula (26), reduces to the following one given first by Langlands [38]

$$u(x,t)=\frac{1}{\sqrt{4\pi Pt^{\alpha}}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left[\frac{(Q+T)t^{(\beta-\alpha)}}{P}\right]^m\times H_{2,3}^{2,1}\left[\frac{x^2}{4Pt^{\alpha}}\,\middle|\,\begin{matrix}(1/2-m,1),(1-\alpha/2+(\beta-\alpha)m,\alpha)\\(0,1)(1/2,1),(1/2,1)\end{matrix}\right]. \tag{42}$$

Furthermore, when $\beta=\gamma=\alpha$, using (A.6) and (A.7), formula (41) reduces to the fundamental solution of the time-fractional diffusion equation first obtained by Mainardi [45,11] in terms of a Wright-type function now known as the M-function [46,47,25,40] and later in terms of the H-function [48,16]

$$u(x,t)=\frac{1}{\sqrt{4\pi\mathcal{D}t^{\alpha}}}H_{1,2}^{2,0}\left[\frac{x^2}{4\mathcal{D}t^{\alpha}}\,\middle|\,\begin{matrix}(1-\alpha/2,\alpha)\\(0,1)(1/2,1)\end{matrix}\right], \tag{43}$$

$$=\frac{1}{2\sqrt{\mathcal{D}t^{\alpha}}}H_{1,1}^{1,0}\left[\frac{|x|}{\sqrt{\mathcal{D}t^{\alpha}}}\,\middle|\,\begin{matrix}(1-\alpha/2,\alpha/2)\\(0,1)\end{matrix}\right], \tag{44}$$

$$=\frac{1}{2\sqrt{\mathcal{D}t^{\alpha}}}M_{\alpha/2}\left(\frac{|x|}{\sqrt{\mathcal{D}t^{\alpha}}}\right), \tag{45}$$

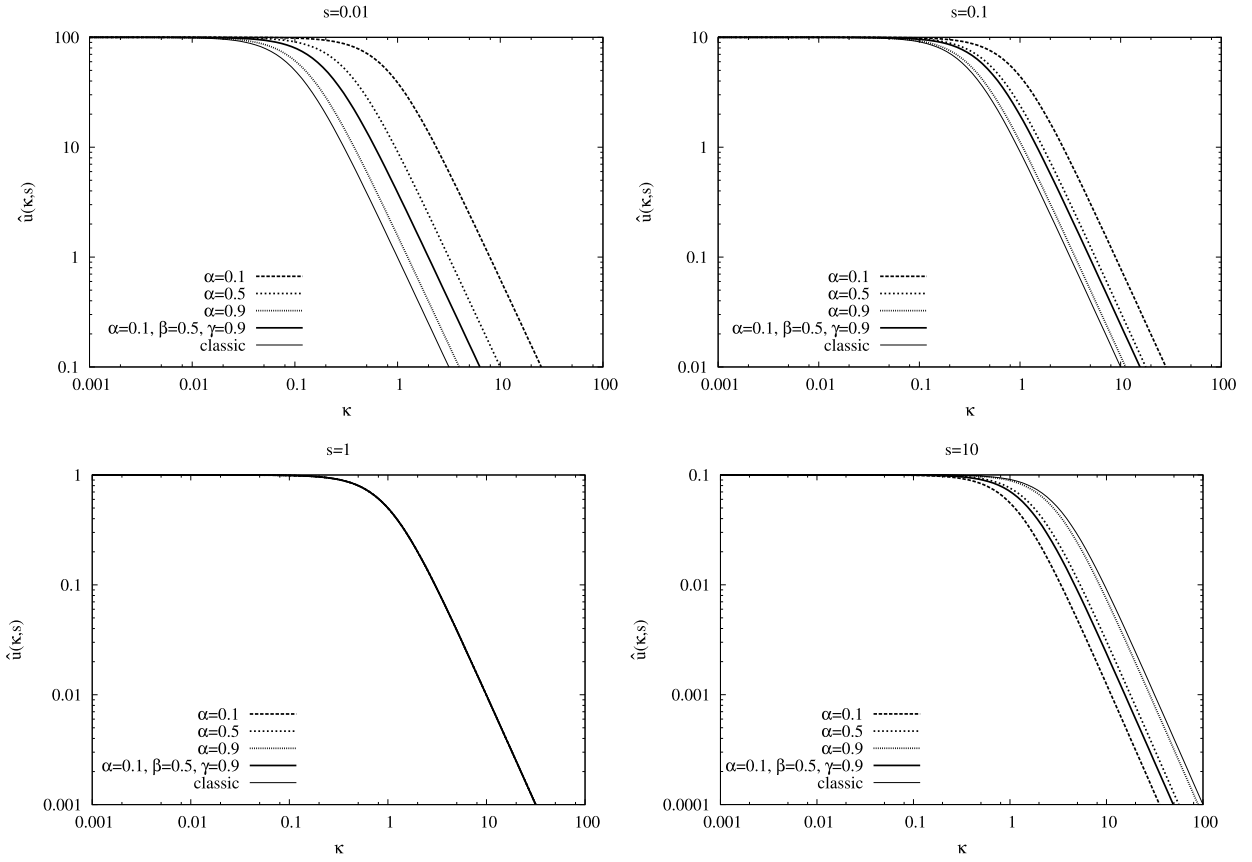

Fig. 2. Comparison of the wavenumber–frequency spectra of triple-order (13) and classic cases with that of the single-order case when $s = 0.01, 0.1, 1$ and 10. In the single-order case $\{P = 1, Q = T = 0, \alpha = 0.1, 0.5, 0.9\}$ and in the triple-order case $\{P = Q = T = 1/3, \alpha = 0.1, \beta = 0.5, \gamma = 0.9\}$.

where $\mathcal{D} = P + Q + T = 1$ acts as the fractional diffusion coefficient. Representation (44) follows from (43) using its Mellin–Barnes integral representation and the duplication formula of Gamma function.

The Fourier–Laplace transform (13) can be seen also as a wavenumber–frequency spectrum of the fundamental solution of the distributed time-fractional accelerating diffusion. For fixed frequencies $s = 0.01, 0.1, 1$ and 10, the spectra of the single, double and triple order are plotted and compared in Figs. 2 and 3 together with the spectrum of the classical diffusion equation (i.e. single order equal to 1). It is possible to observe that, for all frequencies s, at small wavenumbers κ the behaviour is the same while it differs for large wavenumbers. When $s < 1$, in analogy with the results showed in Fig. 1 for relaxation, in the multi-order cases because of the presence of orders near to 1, their spectrum will be closer to that of the classic case than the single-order spectra with order much less than 1. On the other hand, the presence in the multi-order cases of orders much less than 1 leads to their spectrum be less close to the classic one in comparison to the single-order spectrum with order close to 1. The same type of behaviour is observed between triple and double order when the higher orders are compared. When $s = 1$, all spectra collapse to a unique profile as follows from formula (13). For $s > 1$, if the lower order is considered, then the triple order will be further away from the classic spectrum than the single-order cases with orders higher than it. The same is not observed with the double-order spectrum because the lower order taken into account in both cases.

4. Conclusion

In the present paper we have considered the triple-order time-fractional differential equations, with derivative orders less than 1, for modelling both accelerating relaxation and accelerating diffusion, by using Riemann–Liouville fractional differential operator. The corresponding analysis for decelerating relaxation and decelerating diffusion, by using Caputo fractional differential operator, is considered in the companion paper [36].

A new method is outlined. It requires certain properties of the generalized Mittag-Leffler function and the H-function. It is found that the solutions of both time-fractional equations with three time scales are infinite series of H-functions instead of a simple unique function, as in the case of the corresponding single-order time-fractional equations. If we go further for more scales than three, the results will be still obtained in terms of infinite series of H-functions, as can be seen from the derivation of both Green functions (30) and (41).

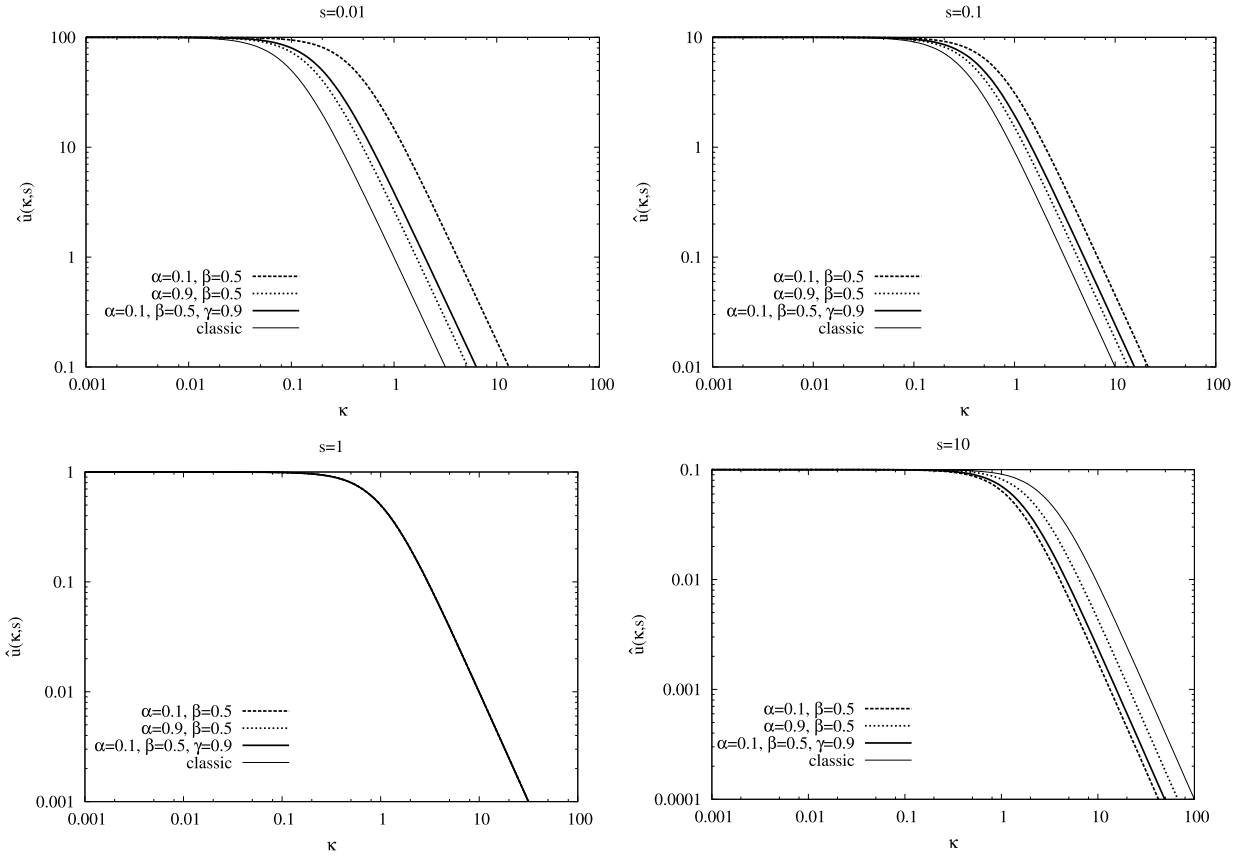

Fig. 3. Comparison of the wavenumber–frequency spectra of triple-order (13) and classic cases with that of the double-order case when $s = 0.01, 0.1, 1$ and 10. In the double-order case $\{P = Q = 1/2, T = 0\}$ with $\{(\alpha = 0.1, \beta = 0.5); (\alpha = 0.5, \beta = 0.9)\}$ and in the triple-order case $\{P = Q = T = 1/3, \alpha = 0.1, \beta = 0.5, \gamma = 0.9\}$.

Thus, the new method provides an elegant generalization of the method introduced by Langlands [38], to study the double-order case of accelerating processes, and it turns out to be a promising technique for future developments in the field, especially with respect to finding exact solutions.

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Appendix A. The *H*-function

The H-function is defined by means of a Mellin–Barnes type integral as follows [49,50]

$$H_{p,q}^{m,n}\left[z\,\middle|\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p)\\ (b_1, B_1), \ldots, (b_q, B_q)\end{matrix}\right] = \frac{1}{2\pi \mathrm{i}} \int_{\Omega} \chi(s) z^{-s}\, \mathrm{d}s, \tag{A.1}$$

with $\mathrm{i} = (-1)^{1/2}$ and

$$\chi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - A_i s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s) \prod_{i=n+1}^{p} \Gamma(a_i + A_i s)}, \tag{A.2}$$

where an empty product is always interpreted as unity, $\{m, n, p, q\} \in \mathbf{N}_0$ with $1 \le m \le q$ and $0 \le n \le p$, $\{A_i,\ B_j\} \in \mathbf{R}^+$ and $\{a_i, b_j\} \in \mathbf{R}$, or $\mathbf{C}$, with $i = 1, \ldots, p$ and $j = 1, \ldots, q$ such that

$$A_i(b_j + k) \neq B_j(a_i - \ell - 1), \quad k, \ell \in \mathbf{N}_0;\ i = 1, \ldots, n;\ j = 1, \ldots, m. \tag{A.3}$$

The poles of the integrand in (A.1) are assumed to be simple.

Properties of the H-function used in the present paper are: [50, formula (1.60)]

$$z^{\sigma} H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1)\dots(a_p,A_p)\\(b_1,B_1)\dots(b_p,B_p)\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1+\sigma A_1,A_1)\dots(a_p+\sigma A_p,A_p)\\(b_1+\sigma B_1,B_1)\dots(b_p+\sigma B_p,B_p)\end{matrix}\right.\right], \quad \sigma \in \mathbf{C}; \tag{A.4}$$

[50, formula (1.88)] for $\omega \neq 0$ and $\eta \neq 0$ complex numbers it holds

$$H_{p,q}^{m,n}\left[\eta\omega\left|\begin{matrix}(a_1,A_1)\dots(a_p,A_p)\\(b_1,B_1)\dots(b_p,B_p)\end{matrix}\right.\right] = \eta^{b_1/B_1}\sum_{r=0}^{\infty}\frac{(1-\eta^{1/B_1})^r}{r!}H_{p,q}^{m,n}\left[\omega\left|\begin{matrix}(a_1,A_1)\dots(a_p+A_p,A_p)\\(b_1+r,B_1)\dots(b_p+B_p,B_p)\end{matrix}\right.\right], \tag{A.5}$$

where η is arbitrary for $m = 1$, while for $m > 1$ the factor η is such that $|\eta^{1/B_1} - 1| < 1$, $\arg(\eta\omega) = B_1 \arg(\eta^{1/B_1}) + \arg(\omega)$ and $|\arg(\eta^{1/B_1})| < \pi/2$; [50, formula (1.90)] for $\omega \neq 0$ and $\eta \neq 0$ complex numbers it holds

$$H_{p,q}^{m,n}\left[\eta\omega\left|\begin{matrix}(a_1,A_1)\dots(a_p,A_p)\\(b_1,B_1)\dots(b_p,B_p)\end{matrix}\right.\right] = \eta^{(a_1-1)/A_1}\sum_{r=0}^{\infty}\frac{(1-\eta^{-1/A_1})^r}{r!}H_{p,q}^{m,n}\left[\omega\left|\begin{matrix}(a_1-r,A_1)\dots(a_p+A_p,A_p)\\(b_1,B_1)\dots(b_p+B_p,B_p)\end{matrix}\right.\right], \tag{A.6}$$

provided $n > 0$, $\mathrm{Re}(\eta^{1/A_1}) > 1/2$, $\arg(\eta\omega) = A_1 \arg(\eta^{1/A_1}) + \arg(\omega)$ and $|\arg(\eta^{1/A_1})| < \pi/2$; [50, formula (1.56)]

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1)\dots(a_p,A_p)\\(b_1,B_1)\dots(b_{p-1},B_{p-1})(a_1,A_1)\end{matrix}\right.\right] = H_{p-1,q-1}^{m,n-1}\left[z\left|\begin{matrix}(a_2,A_2)\dots(a_p,A_p)\\(b_1,B_1)\dots(b_{p-1},B_{p-1})\end{matrix}\right.\right], \tag{A.7}$$

provided $n \geq 1$ and $q > m$.

To conclude we report below the cosine transform of the H-function which is used in the text, see Ref. [51, p. 682 formula (3)] or with further manipulations [50, p. 58 formula (2.50)],

$$\begin{aligned}&\int_0^{\infty} z^{\rho-1}\cos(\kappa z) H_{p,q}^{m,n}\left[\omega z^{\mu}\left|\begin{matrix}(a_1,1),\dots,(a_p,A_p)\\(b_1,1),\dots,(b_q,B_q)\end{matrix}\right.\right]\mathrm{d}z\\ &\quad = (\pi/\kappa^{\rho}) H_{q+1,p+2}^{n+1,m}\left[\frac{\kappa^{\mu}}{\omega}\left|\begin{matrix}(1-b_q,B_q),(1/2+\rho/2,\mu/2)\\(\rho,\mu)(1-a_p,A_p),(1/2+\rho/2,\mu/2)\end{matrix}\right.\right],\end{aligned} \tag{A.8}$$

where $\mathrm{Re}\left[\rho + \mu \min_{1\leq j\leq m}\left(\frac{b_j}{B_j}\right)\right] > 0$, $\mathrm{Re}\left[\rho + \mu \max_{1\leq j\leq n}\left(\frac{a_j-1}{A_j}\right)\right] < 1$, and $\kappa, \mu > 0$, $\rho, \omega \in \mathbf{C}$, $|\arg\omega| < \pi\varphi/2$, $\varphi > 0$, where φ is defined by

$$\varphi = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0.$$

Appendix B. Prabhakar generalization of Mittag-Leffler function

The Mittag-Leffler function is defined by means of the series representation

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \quad \alpha \in \mathbf{C};\ \mathrm{Re}(\alpha) > 0, \tag{B.1}$$

and its generalization is

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \{\alpha,\beta\} \in \mathbf{C};\ \mathrm{Re}(\alpha),\ \mathrm{Re}(\beta) > 0. \tag{B.2}$$

A further generalization was introduced in 1971 by Prabhakar [52] in the form

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \quad \{\alpha,\beta,\gamma\} \in \mathbf{C};\ \mathrm{Re}(\alpha), \mathrm{Re}(\beta), \tag{B.3}$$

where $(\gamma)_n$ is the Pochhammer symbol defined by

$$(\gamma)_0 = 1, \qquad (\gamma)_n = \gamma(\gamma+1)\dots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \quad \text{when } \gamma \neq 0,$$

whenever $\Gamma(\gamma)$ is defined. $E_{\alpha,\beta}^{\gamma}(z)$ is an entire function of order $\rho = [\mathrm{Re}(\alpha)]^{-1}$ and type $\sigma = \frac{1}{\rho}\left[\mathrm{Re}(\alpha)^{\mathrm{Re}(\alpha)}\right]^{-\rho}$, see Ref. [53]. This function was studied earlier by Prabhakar [52], Saxena [54], Kilbas et al. [55] and Saxena and Saigo [56].

It is important to remark here that when γ is a positive integer then (B.3) is related to (B.2). In fact, remembering the formula of the m-derivative of $E_{\alpha,\beta}(z)$, see Ref. [40, p. 246 formula (9.7)],

$$E_{\alpha,\beta}^{(m)} = \sum_{n=0}^{\infty}\frac{(m+n)!}{\Gamma(\alpha n+\alpha m+\beta)}\frac{z^n}{n!},$$

the function $E^{\gamma}_{\alpha,\beta}(z)$ for a positive integer third parameter $\gamma = m + 1$ turns out to be

$$E^{m+1}_{\alpha,\beta}(z) = \frac{1}{m!} E^{(m)}_{\alpha,\beta-\alpha m}(z). \tag{B.4}$$

Some special cases of interest of this function are enumerated below

(i) $E_{\alpha}(z) = E^{1}_{\alpha,1}(z); \qquad E_{\alpha,\beta}(z) = E^{1}_{\alpha,\beta}(z),$

(ii) $\Phi(\gamma, \beta; z) = \Gamma(\beta) E^{\gamma}_{1,\beta}(z),$

where $\Phi(\gamma, \beta; z)$ is the Kummer confluent hypergeometric function [57, p. 248, Eq. (1)]. The Laplace transform of the generalized Mittag-Leffler function is given by [52]

$$\mathcal{L}\left\{z^{\beta-1} E^{\gamma}_{\alpha,\beta}(cz^{\alpha}); s\right\} = s^{-\beta}(1 - cs^{-\alpha})^{-\gamma}, \quad s \in \mathbf{C},\ \mathrm{Re}(s) \tag{B.5}$$

where $\{\alpha, \beta, \gamma, c\} \in \mathbf{C}, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ and $|cs^{-\alpha}| < 1$. Formula (B.5) can be used also for Laplace transform inversion by virtue of a theorem given in the book by Doetsch [58, Section 22]. For $\gamma = 1$ formula (B.5) reduces to

$$\mathcal{L}\left\{z^{\beta-1} E_{\alpha,\beta}(cz^{\alpha}); s\right\} = s^{-\beta}(1 - cs^{-\alpha})^{-1}. \tag{B.6}$$

To conclude, we give the H-function representation of (B.3)

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{matrix}(1-\gamma, 1)\\ (0, 1), (1-\beta, \alpha)\end{matrix}\right], \tag{B.7}$$

which can be checked to give the series (B.3) after application of the residual theorem to the Mellin–Barnes integral representation (A.1) and (A.2).

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