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SURVEY PAPER

NONLINEAR TIME-FRACTIONAL DIFFERENTIAL EQUATIONS IN COMBUSTION SCIENCE

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Abstract

Dedicated to Professor R. Gorenflo on the occasion of his 80th birthday

The application of Fractional Calculus in combustion science to model the evolution in time of the radius of an isolated premixed flame ball is highlighted. Literature equations for premixed flame ball radius are re-derived by a new method that strongly simplifies previous ones. These equations are nonlinear time-fractional differential equations of order 1/2 with a Gaussian underlying diffusion process. Extending the analysis to self-similar anomalous diffusion processes with similarity parameter $\nu/2 > 0$, the evolution equations emerge to be nonlinear time-fractional differential equations of order $1-\nu/2$ with a non-Gaussian underlying diffusion process.

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Key Words and Phrases: time-fractional derivative, nonlinear equation, anomalous diffusion, combustion science, premixed flame ball

1. Introduction

Fractional Calculus is largely adopted as successfully mathematical tool in several different research fields including anomalous diffusion, viscoelasticity, control theory, biology, finance and many others. Especially it turns out to be useful for modelling phenomena in complex systems. In this paper it is highlighted a new scientific field where Fractional Calculus can show

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its powerful. This new field is the combustion science and in particular the determination of the evolution in time of the radius of a premixed flame ball. It is worthnoting to stress since now that in this application fractional differentiation emerges by physical arguments and not by replacing ordinary differential operators with fractional operators.

A flame ball is an isolated three-dimensional combustion spot with spherical symmetry that occurs in a lean premixed mixture when the combustion process is the one-step irreversible chemical reaction $Fresh\ Gas \rightarrow Burnt\ Gas + Heat$. In premixed combustion all reactants are intimately mixed at the molecular level before the combustion is started, while in non-premixed combustion the fuel and the oxidant must be mixed before than combustion can take place.

Stable flame ball was theoretically predicted in 1944 by the Soviet physicist Zeldovich [23] as exact solution to the heat and mass conservation equations that, in spherical geometry with radial coordinate denoted by r, turn out to be

$$\rho\, C_p \left(\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial r} \right) = h\, \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + QW\,, \tag{1}$$

$$\rho \left(\frac{\partial Y_F}{\partial t} + U \frac{\partial Y_F}{\partial r} \right) = \rho D_F\, \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial Y_F}{\partial r} \right) - W\,, \tag{2}$$

where T is the temperature, Y_F the mass fraction of the fresh gas, U the radial velocity, W the chemical rate, Q the heat of reaction, ρ the specific mass of fresh gas, C_p the specific heat at constant pressure, h the heat conductivity, D_F the diffusion coefficient of the fresh gas. From (1) and (2), temperature and mass concentration fields are related by

$$Le\,(T - T_\infty) = \frac{Q}{C_p} \left(1 - \frac{Y_F}{Y_\infty} \right)\,, \tag{3}$$

where the nondimensional number $Le = h/(\rho\, D_F\, C_p)$ is called Lewis number and T_∞ and Y_∞ are the reference values for temperature and mass fraction of fresh gas, respectively.

After transformation (3), and setting without loss of generality $T_\infty = 0$ and $Y_\infty = 1$, equation (1) gives

$$\rho \left(\frac{\partial Y_F}{\partial t} + U \frac{\partial Y_F}{\partial r} \right) = Le\, \rho D_F\, \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial Y_F}{\partial r} \right) - LeW. \tag{4}$$

For large activation energy the chemical source term behaves like a Dirac δ-function at the flame sheet [3] so, when the steady and convection-free case is considered, equations (1) and (2) reduce to

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = 0 \quad \text{and} \quad \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial Y_F}{\partial r} \right) = 0\,, \tag{5}$$

and the solutions are of the form $c_1 + c_2/r$, where c_1 and c_2 are constants. As it has been clearly reviewed by Ronney [20], this form satisfies the requirement that T and Y_F be bounded as $r \to \infty$, while for cylindrical and planar geometry the corresponding forms are $c_1 + c_2 \ln r$ and $c_1 + c_2 r$, respectively, which are obviously unbounded as $r \to \infty$. For this reason theory admits steady flame ball solutions, but not steady "flame cylinder" or steady "flame slab" solutions.

Zeldovich showed that, for an adiabatic flame ball, the temperature at the surface of the flame ball T_* is $T_* = T_{ad}/Le$, where T_{ad} is the adiabatic homogeneous flame temperature. Flame balls can exist if $T_* > T_{ad}$, which means $Le < 1$, because when $T_* < T_{ad}$ flame balls are weaker than plane flames. Then, inside the ball ($r < R$), the temperature profile $T(r)$ is constant and corresponds to the burnt gas temperature and, outside the ball ($r > R$), it decreases depending on the flame ball radius R with the law

$$T(r) = T_* \frac{R}{r}, \quad r > R. \tag{6}$$

What concerns the fresh gas mass fraction, it is null inside the ball and, as it follows from formula (3), it increases outside the ball as

$$Y_F = 1 - \frac{R}{r}, \quad r > R, \tag{7}$$

where $C_p T_{ad}/Q = 1$. This steady state can be realized only if the flame ball radius R is constant in time. Then, the evolution equation for the flame ball radius and the later analysis on the stability of the solution are necessary. Here the problem to derive the evolution equation for $R(t)$ is addressed.

Even if theoretically predicted in 1944, stable flame balls were accidentally experimentally discovered only in 1984, during short-duration drop tower experiments conducted by Ronney and collaborators [18, 19]. They were finally experimentally established in 1998 from space flight experiment conducted on the STS-83/MSL-1 Space Shuttle mission [20]. In fact, a micro-gravity environment is needed to obtain spherical symmetry and to avoid buoyancy-induced extinction of the flame ball.

Evolution equations for the flame ball radius R have been derived [3, 4, 12, 13] and they emerge to be nonlinear fractional differential equations of order 1/2. However all derivations are based on a complex matching of multiple asymptotic expansions. In the present paper a new method to derive the evolution equation for the flame ball radius is shown. It strongly simplifies previous methods and moreover it highlights that considered literature equations are founded on the classical diffusion process which is characterized by a Gaussian probability density function (*pdf*) and a linear growth of the variance of particle displacement. By the same method, further development including self-similar anomalous diffusion processes is performed.

In Section **2** the mathematical preliminaries used in the rest of the text on Fractional Calculus and anomalous diffusion are introduced. In Section **3** the new method is presented for the literature case studies [3, 4, 13] and the corresponding equations are re-derived. In Section **4** the same method is used to formulate an equation for the evolution of the flame ball radius in the general case with a polynomial forcing and anomalous diffusion. Finally, in Section **5** the conclusion and perspective for future developments are discussed.

2. Mathematical preliminaries

This introductory section to Fractional Calculus follows the 1996 CISM lectures by Gorenflo and Mainardi [7], which were partly based on the book on Abel Integral Equations by Gorenflo and Vessella [5] and on the article by Gorenflo and Rutman [6].

Let $f(t)$, with $t > 0$, be a sufficiently well-behaved function, Riemann–Liouville and Caputo fractional derivatives are both based on Riemann-Liouville fractional integral that, when it is of order $\alpha > 0$, is defined as

$$ {}_tJ^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau\,, \quad \alpha > 0\,. \tag{8}$$

The operator ${}_tJ^{\alpha}$ is conventionally the Identity operator when $\alpha = 0$, i.e. ${}_tJ^0 = I$, and it meets the semigroup property

$$ {}_tJ^{\alpha}\, {}_tJ^{\beta} = {}_tJ^{\beta}\, {}_tJ^{\alpha} = {}_tJ^{\alpha+\beta}\,, \quad \alpha, \beta \geq 0\,. \tag{9}$$

The Riemann-Liouville time fractional derivative of order $\mu > 0$ is defined, in analogy with the ordinary derivative, as the operator ${}_tD^{\mu}$ which is the left inverse of the Riemann-Liouville integral of order μ

$$ {}_tD^{\mu}\, {}_tJ^{\mu} = I\,, \quad \mu > 0\,. \tag{10}$$

If m denotes the positive integer such that $m-1 < \mu \leq m$, then from (9) and (10) it follows that ${}_tD^{\mu} f(t) := {}_tD^{m}\, {}_tJ^{m-\mu} f(t)$. Hence for $m-1 < \mu < m$

$$ {}_tD^{\mu} f(t) = \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-\mu)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\mu+1-m}}\, d\tau \right], \tag{11}$$

and ${}_tD^{\mu} f(t) = d^m f(t)/dt^m$ when $\mu = m$.

On the other hand, the fractional derivative of order $\mu > 0$ in the Caputo sense is defined as the operator ${}_tD_*^{\mu}$ such that ${}_tD_*^{\mu} f(t) := {}_tJ^{m-\mu}\, {}_tD^{m} f(t)$. Hence for $m-1 < \mu < m$,

$$ {}_tD_*^{\mu} f(t) = \frac{1}{\Gamma(m-\mu)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\mu+1-m}}\, d\tau \,, \tag{12} $$

and ${}_tD_*^{\mu} f(t) = d^m f(t)/dt^m$ when $\mu = m$. Thus, when the order is not integer the two fractional derivatives mainly differ because the derivative of order m does not generally commute with the fractional integral.

Furthermore, unlike Riemann–Liouville fractional derivative, Caputo fractional derivative satisfies the relevant property of being zero when it is applied to a constant, and, in general, when its order μ is such that $m-1 < \mu \le m$, to any power function of non-negative integer degree less than m.

Moreover, Gorenflo and Mainardi have shown [7] the essential relationship between the two fractional derivatives (when both of them exist), for $m-1 < \mu < m$, which is

$$ {}_tD_*^{\mu} f(t) = {}_tD^{\mu} f(t) - \sum_{n=0}^{m-1} \frac{f^{(n)}(0^+)\, t^{n-\mu}}{\Gamma(n-\mu+1)} \,. \tag{13} $$

The Caputo fractional derivative is a regularization in the time origin of the Riemann–Liouville fractional derivative. From (13) emerges that for its existence all the limiting values $f^{(n)}(0^+) := \lim_{t\to 0^+} f(t)$ are required to be finite for $n = 0, 1, 2, \dots, m-1$. In the special case $f^{(n)}(0^+) = 0$ for $n = 0, 1, \dots, m-1$, the identity between the two fractional derivatives follows.

Besides Riemann–Liouville and Caputo fractional derivative, usually intended as time-fractional derivative, there is also Riesz–Feller fractional derivative that indeed is generally intended for space-fractional derivative. The Riesz–Feller fractional derivative of order α, with $0 < \alpha \le 2$, is defined as [8, 10, 16]

$$ {}_xD_\theta^{\alpha} f(x) = \frac{\Gamma(1+\alpha)}{\pi} \left\{ \sin[(\alpha+\theta)\pi/2] \int_0^{\infty} \frac{f(x+\xi) - f(x)}{\xi^{1+\alpha}}\, d\xi \right. $$
$$ \left. + \sin[(\alpha-\theta)\pi/2] \int_0^{\infty} \frac{f(x-\xi) - f(x)}{\xi^{1+\alpha}}\, d\xi \right\} \,, \tag{14} $$

where $|\theta| \le \min\{\alpha, 2-\alpha\}$ is the asymmetry parameter. For $\theta = 0$ it reduces to the symmetric operator

$$ {}_xD_0^{\alpha} = -\left(-\frac{d^2}{dx^2} \right)^{\alpha/2} \,, \tag{15} $$

from which it follows that ${}_xD_0^{2} f(x) = d^2 f(x)/dx^2$ and

$$ {}_xD_0^{1} f(x) = -\frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^{+\infty} \frac{f(\xi)}{x-\xi}\, d\xi \,, \tag{16} $$

which is related to the Hilbert transform. Other interesting special cases are

$$ {}_xD^1_\theta f(x) = \left[\cos(\theta\pi/2)\,{}_xD^1_0 + \sin(\theta\pi/2)\frac{d}{dx}\right] f(x)\,, \tag{17} $$

and the following extremal cases ${}_xD^1_{\pm 1} f(x) = \pm df(x)/dx$.

Fractional Calculus is emerged to be the most successful tool to model anomalous diffusion [22]. Generally, anomalous diffusion is met in complex media. Models based on fractional differential equations have been proposed in a large number of research fields. In this respect the valuable work by Prof. Gorenflo is here noticed and remarked by reminding some of his most cited papers: References [8, 9, 11].

Let $P(x,t)$ be the *pdf* to find a particle in x at time t. A quite general comprehensive modelling of anomalous diffusion is the space-time fractional diffusion equation [16]

$$ {}_tD^\beta_* P(x,t) = {}_xD^\alpha_\theta P(x,t)\,, \quad x \in \mathcal{R}\,, \quad t \in \mathcal{R}^+_0\,, \tag{18} $$

where the *pdf* interpretation of $P(x,t)$ holds in the following cases [16]: if $0 < \beta \leq 1$ then $0 < \alpha \leq 2$; if $1 < \beta \leq 2$ then $1 < \beta < \alpha \leq 2$. The Mellin–Barnes integral representation of the Green function of (18), i.e. with initial condition $P(x,0) = \delta(x)$, when $|\theta| \leq 2 - \beta$ is (see [16])

$$ \mathcal{G}(x,t) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(s/\alpha)\Gamma(1-s/\alpha)\Gamma(1-s)}{\Gamma(1-(\beta/\alpha)s)\Gamma(\rho s)\Gamma(1-\rho s)}\left(\frac{x}{t^{\beta/\alpha}}\right)^s ds\,, \tag{19} $$

where $\rho = (\alpha - \theta)/(2\alpha)$ and $0 < \sigma < \min\{\alpha, 1\}$.

Special cases are here briefly reminded. When $\beta = 1$ and $0 < \alpha < 2$ (space-fractional diffusion equation) the Green function corresponds to the Lévy class of strictly stable (non-Gaussian) densities which exhibits fat tails with the algebraic decay proportional to $|x|^{-(\alpha+1)}$ and infinite variance. When $\alpha = 2$ and $0 < \beta < 2$ (time-fractional diffusion-wave equation [15]) the Green function belongs to the class of M-Wright function [17] with stretched exponential tails and finite variance proportional to t^β. Finally, when $\beta = 1$ and $\alpha = 2$ the classical diffusion equation is recovered where the Green function is Gaussian and the displacement variance grows linearly in time.

With reference to the following analysis, here it is reminded the value in the origin of the Green function (19) (see [16])

$$ \mathcal{G}(0,t) = \begin{cases} \dfrac{t^{-\beta/\alpha}}{\pi\alpha}\,\dfrac{\Gamma(1/\alpha)\Gamma(1-1/\alpha)}{\Gamma(1-\beta/\alpha)}\,\cos\left(\dfrac{\theta\pi}{2\alpha}\right), & 1 < \alpha \leq 2\,,\ \beta \neq 1\,, \\[2ex] \dfrac{t^{-\beta/\alpha}}{\pi\alpha}\,\Gamma(1/\alpha)\,\cos\left(\dfrac{\theta\pi}{2\alpha}\right), & 0 < \alpha \leq 2\,,\ \beta = 1\,. \end{cases} \tag{20} $$

In general, anomalous diffusion corresponding to different phenomena is described by different fractional differential equations which could be also nonlinear, see e.g. [14]. However, in all cases, if the process is self-similar the Green function has the general form

$$\mathcal{G}(x,t) = \frac{1}{t^{\nu/2}}\,\mathcal{H}\left(\frac{x}{t^{\nu/2}}\right) , \tag{21}$$

where $\nu/2 > 0$ is the similarity parameter and $\mathcal{H}(x/t^{\nu/2})$ is called reduced Green function. In particular, the Green function of the space-time fractional diffusion (19) can be written as

$$\mathcal{G}(x,t) = t^{-\beta/\alpha}\mathcal{H}\left(\frac{x}{t^{\beta/\alpha}}\right) , \tag{22}$$

and the similarity parameter is $\nu/2 = \beta/\alpha$.

3. The evolution equation for the flame ball radius

Let R be at any fixed instant t the radius of the flame ball, then its growing in time is here assumed to be determined by the evolution of the matching interface between an inner kernel ($r < R$), which is the quasi-stationary spherical solution of a Poisson-type equation, and an outer diffusive part ($r > R$), which is the solution of a diffusion equation.

Let ϕ_s be the inner solution and ϕ_d be the outer solution. Then the growing in time of the flame ball radius is determined by a diffusion operator that acts on the inner solution computed on the surface of the flame ball. This means that the source term of the diffusion process is determined by $\phi_s(x,t)\delta(x-R(t))$ and the action of the operator emerges to be a double convolution integral both in space and time with propagating kernel $\mathcal{K}(x,t)$, i.e.

$$R(t) = \mathcal{K}(x,t) * \phi_s(x,t)\delta(x-R(t)) = \phi_d(R,t)\,. \tag{23}$$

This matching method has been suggested to the author by the diffusive formulation discussed in References [1, 2, 21]. Moreover, such diffusive formulation, has been used by Gorenflo and Vessella [5] to study Volterra integral equations.

Consider a flame initiated by a point source energy input which spherically evolves under the action of a radial forcing $\sim 1/r^2$ and radiative heat losses $\sim -\lambda$, then the inner solution ϕ_s is determined as the quasi-stationary solution of the Poisson-type equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial \phi_s}{\partial r}\right] = \frac{2}{r^2} - 12\lambda\,, \tag{24}$$

with boundary condition

$$\left[r^2 \frac{\partial \phi_s}{\partial r}\right]_{r=0} = -2Eq(t)\,, \quad q(0) = 0\,, \tag{25}$$

where $Eq(t)$ is a measure of the energy input with $E > 0$ as intensity and $q(t)$ as temporal variation. The numerical factors on RHS of (24) and (25) are chosen for formal reasons. Finally, the inner solution $\phi_s(r,t)$ turns out to be

$$\phi_s(r,t) = 2\left[\ln r + \frac{E\,q(t)}{r} - \lambda\, r^2\right] = 2\, f(r,t)\,. \tag{26}$$

Assume now that each point of the matching interface is diffused along the one-dimensional axes that ranges from $-\infty$ to $+\infty$ and is aligned with r. Then, the spherical reference system characterized by $r > 0$, which was used to determine the growing of the inner solution ϕ_s, is now abandoned to use a one-dimensional Cartesian axes x, such that $|x| = r$, and the diffusion is modelled with respect to this reference frame. This means that now the flame ball radius is located in $|x| = R$.

Finally, the outer diffusive solution ϕ_d is determined as the solution of a diffusion equation with source term S given by the inner solution computed in the inner-outer interface located at the flame position $R(t)$. In the r-coordinate system $S(r,t) = \phi_s(r,t)\delta(r - R(t)) = 2f(r,t)\delta(r - R(t))$ and in the x-coordinate system $S(x,t) = \phi_s(x,t)\delta(x - R(t)) = 2f(x,t)\delta(x - R(t))$. Hence ϕ_d is the solution of the diffusion equation

$$\frac{\partial \phi_d}{\partial t} = \frac{\partial^2 \phi_d}{\partial x^2} + 2\, f(x,t)\, \delta(x - R(t))\,. \tag{27}$$

The Green function of (27) is the Gaussian density

$$\mathcal{G}(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}\,, \tag{28}$$

which describes a normal diffusion process with linear variance growing, i.e. $\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \mathcal{G}(x,t)\,dx \sim t$. Then the solution of (27) turns out to be the following double convolution integral

$$\phi_d(x,t) = 2\int_{-\infty}^{+\infty} d\eta \int_0^t d\tau\, \mathcal{G}(x-\eta, t-\tau) f(\eta,\tau)\delta(\eta - R(\tau))\,,$$

which, solving the convolution in space, reduces to

$$\phi_d(x,t) = 2\int_0^t \mathcal{G}(x - R(\tau), t-\tau) f(R(\tau),\tau)\,d\tau\,.$$

To conclude, inserting (28) in the above formula, the solution of (27) turns out to be

$$\phi_d(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\mathrm{e}^{-(x-R)^2/(4(t-\tau))}}{\sqrt{t-\tau}} \, f(R(\tau),\tau)\, d\tau \,. \tag{29}$$

Comparing (23) and (29) it emerges that the propagator $\mathcal{K}(x,t)$ turns out to be the Gaussian density (28) and the evolution equation for the flame ball radius follows to be

$$R(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{f(R(\tau),\tau)}{\sqrt{t-\tau}} \, d\tau = {}_tJ^{1/2}[f(R(t),t)] \,, \tag{30}$$

with initial condition $R(0) = 0$, where ${}_tJ^{1/2}$ is the Riemann–Liouville fractional integral of order $1/2$ defined in (8). Applying on both sides of (30) the Riemann–Liouville time-fractional derivative operator ${}_tD^{1/2}$, which is defined in (11), gives ${}_tD^{1/2}R(t) = {}_tD^{1/2}{}_tJ^{1/2}[f(R(t),t)] = f(R(t),t)$, where property (10) is used. After multiplication by $R(t)$, the evolution equation (30) becomes the following nonlinear fractional differential equation

$$R(t)\, {}_tD^{1/2}R(t) = R(t) \ln R(t) + Eq(t) - \lambda R^3(t) \,. \tag{31}$$

Relationship (13) between Riemann–Liouville ${}_tD^{\mu}$ and Caputo ${}_tD_*^{\mu}$ fractional derivatives can be applied in (31). Since the order of fractional derivation is $0 < 1/2 < 1$ and $R(0^+) = 0$, then ${}_tD^{1/2}R(t) = {}_tD_*^{1/2}R(t)$. Finally, in terms of Caputo time-fractional derivative, the evolution equation of the flame radius is

$$R(t)\, {}_tD_*^{1/2}R(t) = R(t) \ln R(t) + Eq(t) - \lambda R^3(t) \,. \tag{32}$$

Equation (32) is the Buckmaster–Joulin–Ronney equation [3, 4] and it reduces to the seminal equation derived by Joulin [13] neglecting heat losses, i.e. $\lambda = 0$.

4. The generalized evolution equation for the flame ball radius

Let $\Phi_s(r,t)$ be the solution, namely the generalized inner solution, of the following Poisson-type equation with a general polynomial forcing

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \Phi_s}{\partial r} \right] = \frac{\mathcal{N}}{r^2} - \mathcal{N}\,\lambda \sum_{i=1}^{n} 6\gamma_i \, r^{\xi_i} \,, \tag{33}$$

with the boundary condition

$$\left[r^2 \frac{\partial \Phi_s}{\partial r} \right]_{r=0} = -\mathcal{N}\, Eq(t) \,, \quad q(0) = 0 \,, \tag{34}$$

where the coefficients in the RHS of (33) and (34) are chosen for the same formal reasons of coefficients in (24) and (25). Then the generalized inner solution $\Phi_s(r,t)$ is

$$\begin{aligned}\Phi_s(r,t) &= \mathcal{N}\left[\ln r + \frac{E\,q(t)}{r} - \lambda r^2 \sum_{i=1}^{n} \frac{6\gamma_i\, r^{\xi_i}}{(\xi_i+3)(\xi_i+2)}\right], \\ &= \mathcal{N}\, F(r,t)\,. \end{aligned} \tag{35}$$

For $n=1$ and setting $\mathcal{N}=2$, $\gamma_1=1$, $\xi_1=0$, previous inner solution (26) is recovered as well as $F(r,t)=f(r,t)$.

As explained in the previous section, also in this case the diffusion process occurs along the axes x aligned with $r=|x|$. So the description of the process moves from the spherical coordinate r to a one-dimensional Cartesian reference frame.

As it has been pointed out at the end of Section **2**, even if different types of anomalous diffusion equation have been proposed in literature, if the evolution equations to model anomalous diffusion admit self-similar solutions then the Green functions emerge to be of the form (21). So the solution of the whole diffusion process with source term $S(x,t)=\Phi_s(x,t)\delta(x-R(t))=\mathcal{N}F(x,t)\delta(x-R(t))$ is given by the double convolution integral

$$\Phi_d(x,t) = \mathcal{N}\int_{-\infty}^{+\infty} d\eta \int_0^t d\tau\, \mathcal{G}(x-\eta, t-\tau) f(\eta,\tau)\delta(\eta - R(\tau))\,,$$

which, solving the convolution in space, reduces to

$$\Phi_d(x,t) = \mathcal{N}\int_0^t \mathcal{G}(x-R(\tau), t-\tau) f(R(\tau),\tau)\, d\tau\,.$$

To conclude, inserting (21) in the above formula, the generalized outer solution turns out to be

$$\Phi_d(x,t) = \mathcal{N}\int_0^t \mathcal{H}\left[\frac{x-R(t)}{(t-\tau)^{\nu/2}}\right] \frac{F(R(\tau),\tau)}{(t-\tau)^{\nu/2}}\, d\tau\,. \tag{36}$$

The generalized evolution equation follows from (23) and (36) from which the propagator $\mathcal{K}(x,t)$ is given by the Green function (21). Setting $\mathcal{M}=\mathcal{N}\mathcal{H}(0)\Gamma(1-\nu/2)$, it turns out to be

$$\begin{aligned} R(t) &= \frac{\mathcal{M}}{\Gamma(1-\nu/2)}\int_0^t \frac{F(R(\tau),\tau)}{(t-\tau)^{\nu/2}}\, d\tau \\ &= \mathcal{M}\, {}_tJ^{1-\nu/2}[F(R(t),t)]\,. \end{aligned} \tag{37}$$

Repeating the same steps done for the non anomalous case, in terms of Riemann–Liouville fractional differential operator equation (37) becomes

$$R(t)\,{}_tD^{1-\nu/2}R(t) = \mathcal{M}\left[R(t)\ln R(t) + Eq(t) - \lambda R^3(t)\sum_{i=1}^{n}\frac{6\gamma_i\,R^{\xi_i}(t)}{(\xi_i+3)(\xi_i+2)}\right], \quad (38)$$

and in terms of Caputo fractional differential operator

$$R(t)\,{}_tD_*^{1-\nu/2}R(t) = \mathcal{M}\left[R(t)\ln R(t) + Eq(t) - \lambda R^3(t)\sum_{i=1}^{n}\frac{6\gamma_i\,R^{\xi_i}(t)}{(\xi_i+3)(\xi_i+2)}\right]. \quad (39)$$

The coefficient $\mathcal{M}$ depends on the value of the reduced Green function when $x = 0$. In particular, in the case of space-time fractional diffusion reviewed in Section **2**, using (20), $\mathcal{M}$ turns out to be

$$\mathcal{M} = \frac{\mathcal{N}}{\pi\alpha}\,\Gamma(1/\alpha)\Gamma(1-1/\alpha)\cos\left(\frac{\theta\pi}{2\alpha}\right). \quad (40)$$

It is worth to remark that the value of the coefficient $\mathcal{M}$ is fully determined by the characteristics of the space differentiation. In the normal diffusion case $\alpha = 2$, $\theta = 0$ and then $\mathcal{M} = \mathcal{N}/2$, moreover $\beta = 1$ so that $\nu = 1$ and from the reduced Green function $\mathcal{H}(0) = 1/(2\sqrt{\pi})$. Stating $\mathcal{N} = 2$ when $n = 1$, if $\gamma_1 = 1$ and $\xi_1 = 0$ then the generalized evolution equation (39) reduces to Buckmaster–Joulin–Ronney equation (32), and to Joulin equation setting also $\lambda = 0$; if $\gamma_1 = 1$ and $\xi_1 = -1$ it reduces to

$$R(t)\,{}_tD_*^{1/2}R(t) = R(t)\ln R(t) + Eq(t) - 3\lambda R^2(t)\,, \quad (41)$$

which has been derived by Guyonne and Noble [12] on the basis of the linearized Eddington equation for radiative field.

5. Conclusion

In the present paper a new application of Fractional Calculus is highlighted in the field of combustion science to model the evolution in time of the radius of a premixed flame ball. This process emerges to be described by a nonlinear time-fractional differential equation. When the underlying diffusion is assumed Gaussian then the equation is of order 1/2, as derived in literature, when the underlying diffusion is a self-similar anomalous diffusion process with similarity parameter $\nu/2$ then the equation is of order $1-\nu/2$.

The main remarkable aspect of this new application field is that in literature derivation, as well as in the method here shown, the fractional differential operators emerge by a physically sound depiction of the process and not by substitution of ordinary differential operators with the fractional counterparts. The shown method is based on the idea to split the flame ball in two components: the inner kernel, which is driven by a Poisson-type equation with a general polynomial forcing term, and the outer part, which is driven by a generalized anomalous diffusion process. The evolution equation for the radius of the flame ball is determined as the evolution equation for the interface that matches the solution of the inner spherical kernel and the solution of the outer diffusive part. This method strongly simplifies and generalizes previous derivations and exposes that they are based on Gaussian diffusion. Literature equations [3, 12, 13] are recovered when the forcing and the diffusion process are appropriately chosen.

To conclude, due to its clearness and simple derivation, the shown method can be a useful tool to further development and advance in the research on this topic helping to overcame the difficulties that the current models meet. In fact, the mathematical simplicity of equation foundation can suggest new promising way to find analytical and numerical solution, solution properties as well as to analyse solution stability which is of paramount importance for establishing the experimental configuration to observe the steady flame ball originally predicted by Zeldovich.

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